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Generalized Kaup–Kupershmidt solitons and other solitary wave solutions of the higher order KdV equations

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Abstract
New types of exact explicit solitary wave solutions of the higher order KdV equations are identified by applying a direct method designed specifically for constructing solitary wave solutions of evolution equations. The first type is the 'generalized Kaup–Kupershmidt' (GKK) solitary waves, which unify the structures of the sech^2 KdV-like soliton and the Kaup–Kupershmidt soliton and also provide solutions of some other equations. One of those equations is found to possess the multi-soliton solutions which makes it a good candidate for being integrable in terms of the GKK solitons. Another type of solutions of the higher order KdV equations identified by applying the method represents the steady-state localized structures. The variety of equations possessing such solutions includes the integrable (in terms of the sech^2 solitons) Sawada–Kotera equation which thus appears to provide the localized solutions of two types.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The Korteweg–de Vries (KdV) equation arises in many physical contexts as an equation governing weakly nonlinear long waves when nonlinearity and dispersion are in balance at leading order. If higher order nonlinear and dispersive effects are of interest, then the asymptotic expansion can be extended to the next order in the wave amplitude which leads to the higher order KdV equations. To date, a plethora of results exists concerning behavior of the solitary wave solutions of the higher order KdV equations, most frequently obtained through the use of perturbation (asymptotic) methods. The results obtained this way rely on the existence of the exact solutions in the form of the KdV type solitons

$$u(x,t) = a\ \mathrm{sech}^2 k(x - Vt). \tag{1}$$

Correspondingly asymptotic procedures start either from the KdV equation

$$u_t + 6uu_x + u_{3x} = 0 \tag{2}$$

or from the integrable fifth-order equation, which represents a combination of the KdV equation and its first commuting flow, as follows:

$$u_t + 6uu_x + u_{3x} + \alpha(u_{5x} + 10uu_{3x} + 20u_x u_{2x} + 30u^2 u_x) = 0. \tag{3}$$

where subscripts of the form 'nx' denote the derivatives of the order n with respect to x. Solutions of equation (3) are the KdV solitons with the same amplitude $a = 2k^2$ and the updated wave velocity $V = 4k^2 + \alpha 16k^4$.

The sech^2 solitary wave solutions seem to be ubiquitous for the KdV-like equations, and frequently the problem of the existence of the solitary wave solutions is identified with the possibility of having the sech^2 solutions (see e.g. [1]). Indeed, almost all the KdV-like integrable equations possess such solutions. Among the fifth-order KdV equations there are, in addition to (3), two pure fifth-order equations (not including the KdV flow but including terms having the same structure as the 'perturbation' terms in the perturbed KdV equations) which are known to be integrable in terms of the sech^2 solitons. The first equation is one that belongs to the completely integrable hierarchy of higher order KdV equations (sometimes called the Lax equation [2]):

$$u_t + u_{5x} + 10uu_{3x} + 20u_x u_{2x} + 30u^2 u_x = 0 \tag{4}$$

and the second is the Sawada–Kotera (SK) equation [3] (sometimes called the Caudrey–Dodd–Gibbon equation [4])

$$u_t + u_{5x} + \beta uu_{3x} + \beta u_x u_{2x} + \frac{\beta^2}{5} u^2 u_x = 0. \tag{5}$$

Note that in the last equation β may be scaled to any value (see section 2.1). The KdV–SK equation which represents a combination of the KdV equation and the SK differential polynomial specified to $\beta = 15$ as

$$u_t + 6uu_x + u_{3x} + \alpha(u_{5x} + 15uu_{3x} + 15u_x u_{2x} + 45u^2 u_x) = 0 \tag{6}$$

is not known to be integrable but has two-soliton and three-soliton solutions of the sech^2 form [5].

The only exception in this sense is the integrable Kaup–Kupershmidt (KK) equation [6, 7]

$$u_t + u_{5x} + \beta uu_{3x} + \frac{5}{2}\beta u_x u_{2x} + \frac{\beta^2}{5} u^2 u_x = 0. \tag{7}$$

Solitary wave solutions of the KK equation are frequently called 'anomalous' soliton solutions [1, 8, 9] to emphasize the difference from the common 'regular' sech^2 solitons. These 'anomalous' solitary wave solutions have the form

$$u(x,t) = \frac{60}{\beta} k^2 \frac{2\cosh 2k\xi + 1}{(\cosh 2k\xi + 2)^2}, \qquad \xi = x - 16k^4 t. \tag{8}$$

As distinct from the case of the sech^2 solitons, no other fifth-order equations admitting solutions of form (8) have been found. The KK equation has been widely discussed in the literature, mostly in connection with the SK equation (e.g. [8, 10]) and in efforts to explain the perplexing form of the KK solitons (e.g. [9]).

One of the main results of the present paper is identifying a new type of the exact solitary wave solutions of the higher order KdV equations which includes the sech^2 solitons and the Kaup–Kupershmidt solitons as particular cases. In what follows, solutions of this type are named the 'generalized Kaup–Kupershmidt' (GKK) solitary waves. Among the higher order KdV equations possessing such solutions there is one which provides the GKK solitary waves with a continuous spectrum of the wave velocity. The equation is

$$u_t + 6uu_x + u_{3x} + \alpha\left(u_{5x} + 15uu_{3x} + \tfrac{75}{2}u_x u_{2x} + 45u^2 u_x\right) = 0, \tag{9}$$

and its solutions have the form

$$u(x,t) = 4k^2 \frac{Q\sqrt{\frac{1+20\alpha k^2}{1+5\alpha k^2}}\cosh 2k\xi + 1}{\left(\cosh 2k\xi + Q\sqrt{\frac{1+20\alpha k^2}{1+5\alpha k^2}}\right)^2}, \qquad \xi = x - Vt, \qquad V = 4k^2 + 16\alpha k^4, \tag{10}$$

where $Q = \pm 1$ (with $Q = -1$ allowed only in the case of $\alpha < 0$, see section 3.2).

Equation (9) and its solutions (10) possess several remarkable features. First, as distinct from the case of (3), not only the wave velocity update, as compared with the KdV solitons ($\alpha = 0$), occurs for $\alpha \neq 0$ but also the form of the solution changes essentially. Second, solutions of (9) may be solitary waves of different shapes dependent on the relations between parameters (see more details in section 3.2). Third, solutions (10) include the sech^2 solitons and the KK solitons as the long-wave and the short-wave limits respectively. The KK equation and its solutions loose their isolated and 'anomalous' nature this way. Next, equation (9) possesses multi-soliton solutions that are derived precisely in the same way as those for the pure KK equation which makes it plausible that equation (9) is integrable in terms of the GKK solitons. Thus, there appears one more starting-point for the asymptotic analysis of the higher order KdV equations.

The GKK solitary waves have been identified by applying a direct method designed specifically for finding solitary wave solutions of the evolution equations. In the present paper, the method is applied to the scalar fifth-order equation

$$u_t + 2\hat{q}uu_x + \hat{\mu}u_{3x} + \hat{\alpha}u_{5x} + \hat{\beta}uu_{3x} + \hat{\delta}u_x u_{2x} + 3\hat{r}u^2 u_x = 0, \tag{11}$$

This equation represents a natural extension of the KdV equation to the next order in that the 'perturbation' part includes all possible terms maintaining the balance $\partial_x^2 \sim u$, and being $O(u)$ with respect to the KdV terms. In addition to (10), some other new exact solutions of the equations of type (11) have been found using the method. In particular, new types of solitary wave solutions of the SK equation and the KdV–SK equation have been identified.

Before proceeding, the following remarks are necessary. There exists a one-to-one mapping between the KdV–KK equation (9) and the KK equation (7) specified to $\beta = 15$ so that the transformation

$$u(x,t) = -\frac{1}{15\alpha} + U(X,T), \qquad X = x + \frac{1}{5\alpha}t, \qquad T = \alpha t \tag{12}$$

relates solutions $u(x, t)$ of equation (9) and solutions $U(X, T)$ of the equation

$$U_T + U_{5X} + 15UU_{3X} + \tfrac{75}{2}U_X U_{2X} + 45U^2 U_X = 0. \tag{13}$$

Thus, it is possible to address the result represented by equation (10) as a new type of solutions of the KK equation (13):

$$U(X,T) = \frac{1}{15\alpha} + 4k^2 \frac{Q\sqrt{\frac{1+20\alpha k^2}{1+5\alpha k^2}}\cosh 2k\xi + 1}{\left(\cosh 2k\xi + Q\sqrt{\frac{1+20\alpha k^2}{1+5\alpha k^2}}\right)^2}, \qquad \xi = X - \tilde{V}T, \qquad \tilde{V} = \frac{1+20k^2\alpha + 80k^4\alpha^2}{5\alpha^2}, \tag{14}$$

where α is an arbitrary parameter. Solution (14) may be considered to be more general than (8), in a sense, and including the latter as a limit for $\alpha \to \infty$.

Due to the fact that solution (14) contains a constant part $1/(15\alpha)$, the function $U(X, T)$ does not vanish at infinity but tends to this constant value. Although solitary wave solutions which do not satisfy conditions of vanishing at infinity appear in the literature (e.g. [11, 12]), such a behavior does not seem to be physically meaningful in the context of physical processes described by the original equations. Therefore, it is difficult to attribute physical meaning to such solutions—especially if the constant part depends on the wave number as it does in the solutions of [11] and [12]. Bearing in mind the preceding remarks, we now may be more precise in defining the solitary wave solutions considered in the paper as those possessing the property to vanish at infinity. From this perspective, solution (14) of the KK equation describes a solitary wave only in the limit of $\alpha \to \infty$ when it becomes the KK solitary wave and so the KK equation (13) does not provide the GKK solitary wave solutions (10). From another perspective, if one has solution (14) defined, then one can search for equations providing physically significant solutions without the constant part ('truly' solitary waves), and such a search will result in defining transformation (13) and the KdV–KK equation (9).

The paper is organized as follows. In the next section, the method is described; rescaling possibilities for equation (11) are discussed before applying the method. Solutions obtainable using the method are presented and discussed in section 3. In section 4, some remarks on the results and further perspectives of developing the method are furnished.

2. Method

2.1. Scalings

The number of parameters in equation (11) can be reduced using the scaling freedoms. The cases of $\hat{q} \neq 0$ and $\hat{\mu} \neq 0$, when the KdV flow is included, and of $\hat{q} = \hat{\mu} = 0$ should be treated separately (the cases when only one of the parameters $\hat{q}$ and $\hat{\mu}$ vanishes and the case of $\hat{\alpha} = 0$ are not considered). In the case of $\hat{q} \neq 0$ and $\hat{\mu} \neq 0$, the scales for u and t are defined such that the KdV equations were in its standard form (2). The freedom in the choice of the scale for x is used to fix the coefficient of u_{5x} to either 1 or -1 dependent on the sign of the ratio $\hat{\mu}/\hat{\alpha}$. Thus, in the case of $\hat{q} \neq 0$ and $\hat{\mu} \neq 0$, the equation under consideration is

$$u_t + 6uu_x + u_{3x} + \alpha u_{5x} + \beta uu_{3x} + \delta u_x u_{2x} + 3ru^2 u_x = 0, \tag{15}$$

which, as a matter of fact, represents a four-parameter family of equations with $\alpha = \pm 1$ and arbitrary β, δ and r. (Of course, if it were intended to implement the asymptotic analysis of solutions of (11), the scaling should be such that the small parameter appeared explicitly in the equation.) In the case of $\hat{q} = \hat{\mu} = 0$, the coefficient of the highest derivative is made equal to one by a proper choice of scale for t and, although the two scales for u and x remain free, only one more freedom exists since these two scales enter in the combination $u^*(x^*)^2$.

It is convenient to leave this scale unspecified to facilitate adherence to the forms of some equations used in the literature. Thus, in the case of $\hat{q} = \hat{\mu} = 0$ the equation studied is

$$u_t + u_{5x} + \beta u u_{3x} + \delta u_x u_{2x} + 3 r u^2 u_x = 0, \tag{16}$$

and the equivalent equation might be

$$u_t + u_{5x} + (S\beta) u u_{3x} + (S\delta) u_x u_{2x} + 3(S^2 r) u^2 u_x = 0, \tag{17}$$

where $S = u^*(x^*)^2$ is an arbitrary scale. Thus, (16) represents a two-parameter (not a three-parameter) family of equations. Correspondingly, equations (5) and (7) represent single equations (not one-parameter families) since β can be scaled to any value.

2.2. *Outline of the method*

The specific feature of the direct method applied in this paper is that the 'potential' $p = \int u \, \mathrm{d}x$ is used as an independent variable. In the simplest case, when u is sought to be a function of one variable p, the solution 'Ansatz' has the form

$$u(x,t) = f(p(x,t)), \qquad p(x,t) = \int u(x,t) \, \mathrm{d}x \tag{18}$$

$$p_x = f(p). \tag{19}$$

Note that the choice of limits of integration in the definition of p in (18) is of no importance as it does not influence the solution for $u = p_x$. The method with the solution form (18) is a proper tool if the calculations are aimed at defining the solitary wave solutions. To have solutions in the form of the solitary wave solutions in the original variables, the function $f(p)$ in the 'Ansatz' (18) should have (at least) two real roots. For example, the sech^2 solitary wave solution (1) in these variables takes the form

$$u = a - \frac{k^2 p^2}{a}, \tag{20}$$

and the 'anomalous' solitary wave solution (8) of the KK equation becomes

$$u = -\tfrac{2}{3}(3p^2 + k^2) + \tfrac{4}{3} k \sqrt{3p^2 + k^2}. \tag{21}$$

The potential p in (20) and (21) is defined such that the roots of $f(p)$ were symmetric: $u = 0$ for $p = \pm m/2$.

The forms $f(p)$ in (20) and (21) suggest using one which unifies and somewhat generalizes those two as follows

$$f(p) = AR^2 + BR + C, \qquad R = \sqrt{p^2 + K}, \tag{22}$$

where A, B, C and K are constants. To apply the method one has first to make transformation of variables $(x,t) \to (p(x,t),t) \to (R(x,t),t)$ in the equation under consideration. Then upon substituting form (22) into the transformed equation, one gets a polynomial in R and determines the constants A, B, C and K from conditions of vanishing the monomial coefficients. Having the constants defined, the first-order partial differential equation (19) for $p(x, t)$ can be solved. If u is assumed to be a function of one variable p, as it is in (18), solving equation (19) reduces to one integration while the constant of integration is assigned to be a function of t. This function is to be determined by substitution of the solution $p(x, t)$ of equation (19) into the integrated form of the equation under consideration, and then the function $u(x,t) = p_x(x,t)$ is calculated. Below the solutions obtained by applying the method are presented and discussed.

3. Solutions

3.1. Equations providing the sech^2 *solitary wave solutions*

First of all note that specifying Ansatz (22) to $K = 0$ or $B = 0$ leads to the form corresponding to the sech^2 solitary waves and so applying the method with this specific Ansatz yields such solutions. The fifth-order equations admiting the sech^2 solitary wave solutions were studied in [1] and our method produces results coinciding with those obtained in [1]. It is found that there are equations admiting the sech^2 solutions for a range of wave speeds and equations admiting such solutions for a unique or precisely two possible wave speeds. For the purpose of comparison of the ranges of equation parameters corresponding to the new solutions found in the present paper and those corresponding to the sech^2 solutions, some results of [1] (in somewhat different form) are presented below.

Equations of type (15) admiting the sech^2 solitary wave solutions with continuous spectrum of k are members of two one–parameter families corresponding to $\alpha = 1$ and $\alpha = -1$ which are defined by the relations

$$\delta = 30\alpha - \beta, \qquad r = \beta \tag{23}$$

with the solution parameters $a = 2k^2$ and $V = 4k^2 + \alpha 16k^4$. The integrable equations (3) are included into (23) as a particular case.

Equations of type (16) admiting the sech^2 solitary wave solutions with continuous spectrum of k form a two-parameter family defined by

$$r = \frac{\beta(\beta + \delta)}{30} \tag{24}$$

with $a = 60k^2/(\beta + \delta)$ and $V = 16k^4$, which includes the integrable equations (4) and (5). However, this is in fact a one-parameter family since the scaling freedom related to the scale S in (17) remains unused. This scale may be specified such that the amplitude a of the wave (1) were equal to $2k^2$ as it is for the KdV solitons and for the family (23). This requirement defines the scale as $S = 30/(\beta+\delta)$ and introducing this scale into (17) with r defined as in (24) yields relations between rescaled coefficients coinciding with those of (23) for $\alpha = 1$. Note that such a rescaling based on the requirement that a be equal to $2k^2$ specifies the value of β in the SK equation (5) to $\beta = 15$. Among the KdV–SK equations representing a combination of the KdV equation and the SK differential polynomials with different β, only equation (6) specified to this 'distinguished' value $\beta = 15$ admits the sech^2 solitary wave solutions.

3.2. Generalized Kaup–Kupershmidt solitary waves

Applying the method with Ansatz (22) for $K \neq 0$ and $B \neq 0$ to equations (15) and (16) yields (if solutions not vanishing at infinity and solutions with singularities are excluded) two types of solutions, one of which is considered here and another one in the next subsection. The first type is the generalized KK (GKK) solitary waves:

$$u(x, t) = U\frac{M\cosh 2k\xi + 1}{(\cosh 2k\xi + M)^2}, \qquad \xi = x - Vt, \tag{25}$$

where $M > -1$. The remarkable feature of this form is that it unifies the KK-solitons (8) and the sech^2-solitons (1); the former corresponds to $M = 2$ and the latter to either $M = 1$ (with the wave number k) or $M = 0$ (with the wave number $2k$). Among equations of type (16), only the KK equation (7) admits solutions of form (25) with $M \neq 1, 0$, the KK solitons (8). For equations of type (15) including the KdV flow, the following three cases are identified.

The first case is equation (9) which is a combination of the KdV equation with the KK differential polynomial specified to the value $\beta = 15$; it, as a matter of fact, represents two equations with $\alpha = 1$ and $\alpha = -1$. Solutions of (9) have form (10).

The second case is one of the KdV–SK equations (6) corresponding to $\alpha = -1$:

$$u_t + 6uu_x + u_{3x} - (u_{5x} + 15uu_{3x} + 15u_x u_{2x} + 45u^2 u_x) = 0. \tag{26}$$

It admits the GKK solitary wave solutions (25) with a continuous spectrum of the parameter M but with unique values of the wave number $k = 1/(2\sqrt{5})$ and the wave velocity $V = 4/25$:

$$u(x,t) = \frac{2}{5}\frac{M\cosh\frac{1}{\sqrt{5}}\xi + 1}{\left(\cosh\frac{1}{\sqrt{5}}\xi + M\right)^2}, \qquad \xi = x - \frac{4}{25}t. \tag{27}$$

It is worth remarking that equation (26) provides also the sech^2 solitary waves and thus possesses two types of solitary wave solutions with a continuous spectrum of one of the solution parameters.

The third case is a three-parameter family of equations defined by

$$r = \frac{(4\beta - \delta)(\beta + 2\delta)}{135\alpha}, \tag{28}$$

which admits solutions of the form (25) for specific values of k and other solution parameters:

$$U = \frac{18(\beta + 2\delta - 90\alpha)}{5\beta^2 + 8\beta\delta - 4\delta^2}, \qquad M^2 = \frac{2(\beta - 15\alpha)(\beta + 2\delta)}{3\beta^2 + 30\alpha\delta + 5\beta\delta - 2\delta^2 - 120\alpha\beta}$$
$$k^2 = \frac{\beta + 2\delta - 90\alpha}{20\alpha(5\beta - 2\delta)}, \qquad V = 4k^2 + 16\alpha k^4. \tag{29}$$

In the relations (29), the cases of $\beta = \delta = 15$ and $\delta = 5\beta/2$ must be excluded in order to not have singularities; the case of $\delta = 45\alpha - \beta/2$ should also be excluded as it corresponds to $M = 1$. Then the regions in the equation parameter space corresponding to solutions defined by (25) and (29) do not overlap with (23) corresponding to the sech^2 solitary wave solutions. A particular case of solutions (29) can be obtained from the solution identified in [12] if, in that solution, the wave number is specified to vanish a physically inconsistent constant part of the solution.

Solitary wave solutions of the form (25) are well defined for any $M > -1$ and may have different shapes dependent on the value of M. For $0 \leqslant M \leqslant 2$, their shape is qualitatively similar to that of the common solitons. In this range of M, including $M = 0$ (the sech^2 solitons with the wave number $2k$), $M = 1$ (the sech^2 solitons with the wave number k) and $M = 2$ (the KK solitons), the function $u(\xi)$ has one extremum (maximum) at $\xi = 0$. For $-1 < M < 0$, in addition to the maximum at $\xi = 0$, there appear two minima symmetric to $\xi = 0$. For $2 < M < \infty$, the maximum at $\xi = 0$ changes to a minimum and there appear two additional maxima symmetric to $\xi = 0$—double-hump soliton. Variation of shape of the generalized KK solitary waves with M is shown in figure 1 (values of U and k are chosen as in the solution (27) of equation (26)). The letters (a), (b) and (c) denoting the characteristic shapes for the intervals $-1 < M < 0$, $0 \leqslant M \leqslant 2$ and $2 < M < \infty$ respectively are used for identifying the corresponding shapes and intervals in the following discussion.

For solutions (10) of equations (9), the value of M, and so the shape of the wave, depends on the parameters α, k and Q. In the case of the equation with $\alpha = 1$, only $Q = 1$ is allowed to avoid singularities and values of M are within the interval (b) for any k. Therefore, all solutions of equation (9) with $\alpha = 1$ are solitary waves of a common shape with one maximum which, in the limits of small k and large k, adhere respectively to the sech^2 and KK solitons.

In the case of equation with $\alpha = -1$, the situation is more complicated. First, there exists a gap in the spectrum of k in which the solution does not exist. If $Q = 1$, the solution is well

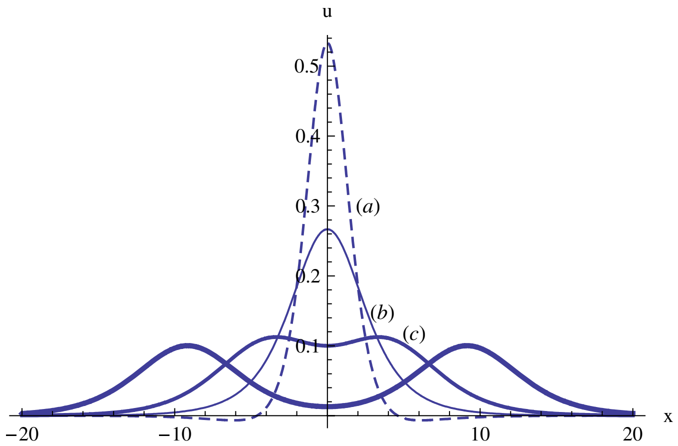

Figure 1. Possible shapes of the GKK solitary waves as defined by (25) plotted for $U = 2/5$, $k = 1/(2\sqrt{5})$, $t = 0$ and different M: dashed for $M = -0.25$, solid for $M = 0.5; 3; 30$ with thickness increasing with M. The letters (a), (b) and (c) are used to denote the characteristic shapes for the intervals $-1 < M < 0$, $0 \leqslant M \leqslant 2$ and $2 < M < \infty$ when respectively the function $u(\xi)$ has one maximum and two minima (a), one maximum (b), and one minimum and two maxima (c).

defined for $0 < k < 1/(2\sqrt{5})$ and $k > 1/\sqrt{5}$, and if $Q = -1$, it is well defined only for $0 < k < 1/(2\sqrt{5})$. Thus, equation (9) with $\alpha = -1$, within the interval $0 < k < 1/(2\sqrt{5})$, possesses solutions of types (a) ($Q = -1$) and (b) ($Q = 1$) and, in the range $1/\sqrt{5} < k < \infty$ for $Q = 1$, possesses solutions of type (c), double-hump solitons. The wave velocity V defined as in (10) may change its sign so that the solution describes the right-running wave for $k < 0.5$ and the left-running wave for $k > 0.5$.

The GKK solitary waves of the form (25) have also been found in [13] as solutions of the sixth-order (and second order in time) equation which is named by the authors as the bidirectional Kaup–Kupershmidt (bKK) equation. Note that, in [13]; the bKK equation is written as a fifth-order equation with a nonlocal term. The multi-soliton solutions of the bKK equation have been studied in [14]. As distinct from the GKK solitary waves described in the present paper with continuous spectrum of the parameter M, for solutions of [13] only two values of the parameter M of (25) are allowed and attributed to the right- and left-running waves which have the shapes (b) regular solitons and (c) double-hump solitons respectively. Thus, the double-hump solitons appear among both solutions found in the present paper and those of [13], but solutions of type (a) identified in the present paper for the KdV-KK and KdV-SK equations do not arise as solutions of the bKK equation of [13]. For both the KdV–KK equation (9) and the bKK equation of [13] solutions describing both right-running and left-running waves are possible. It is worth emphasizing here that the solitary waves (10) arise as solutions of the *unidirectional* equation. Thus, the bidirectional formulation is not obligatorily required to have the GKK solitary waves which can propagate in both directions. The relation between the direction of propagation of the wave and its shape for solutions of the KdV–KK equation (9) is not as simple as for the solitary waves of the bKK equation of [13]. While, similar to [13], the GKK left-running waves may be only of the shape (c), double-hump solitons, as distinct from [13], the right-running waves of all three shapes are possible. In particular, there exist the double-hump waves propagating in both directions: right-running for ($1/\sqrt{5} < k < 0.5$) and left-running for $k > 0.5$.

A remarkable property of equation (9) is that multi-soliton solutions can be derived using Hirota's method modified due to Hereman and Nuseir [15]. The existence of multi-soliton solutions of equation (9), which are derived precisely in the same way as for the pure KK equation, makes it plausible that equation (9) is integrable in terms of the GKK solitons.

The two-soliton solution of equation (9) may be represented in the form

$$\begin{aligned}
&u = \frac{\partial^2 \ln F}{\partial x^2};\\
&F = 1 + q_{11}e^{Z_1} + q_{12}e^{Z_2} + q_{21}e^{2Z_1} + q_{22}e^{2Z_2} + q_{23}e^{Z_1+Z_2}\\
&\qquad + q_{31}e^{2Z_1+Z_2} + q_{32}e^{Z_1+2Z_2} + q_4e^{2Z_1+2Z_2};\\
&Z_1 = 2k_1\left(x - \left(4k_1^2 + 16\alpha k_1^4\right)t - \phi_1\right), \qquad Z_2 = 2k_2\left(x - \left(4k_2^2 + 16\alpha k_2^4\right)t - \phi_2\right);\\
&q_{11} = 2M_1, \qquad q_{12} = 2M_2, \qquad q_{21} = 1, \qquad q_{22} = 1, \qquad q_{23} = \frac{4M_1M_2a}{b},\\
&q_{31} = \frac{2M_2c}{b}, \qquad q_{32} = \frac{2M_1c}{b}, \qquad q_4 = \frac{c^2}{b^2};\\
&M_1 = Q_1\sqrt{\frac{1+20\alpha k_1^2}{1+5\alpha k_1^2}}, \qquad M_2 = Q_2\sqrt{\frac{1+20\alpha k_2^2}{1+5\alpha k_2^2}}; \qquad Q_1 = \pm 1, \qquad Q_2 = \pm 1;\\
&a = 3\left(k_1^2 + k_2^2\right) + 10\alpha\left(2k_1^4 - k_1^2k_2^2 + 2k_2^4\right), \qquad b = (k_1+k_2)^2\left(3 + 20\alpha\left(k_1^2 + k_1k_2 + k_2^2\right)\right),\\
&c = (k_1-k_2)^2\left(3 + 20\alpha\left(k_1^2 - k_1k_2 + k_2^2\right)\right).
\end{aligned} \tag{30}$$

Note that, due to simplifications made in the solution, it can also be applied to a singular case when for one of the solitons (say number two), $M_2 = 0$ $\left(1 + 20\alpha k_2^2 = 0\right)$ which happens for $\alpha = -1$ and $k_2 = 1/(2\sqrt{5})$. (In the initial form of the solution, the coefficient $q_{12} = 2M_2$ appears as a multiple of e^{Z_2} in all the places but simplifications result in that the factor $1+20\alpha k_2^2$ is compensated in some terms.) In this singular case which corresponds to interaction of the GKK soliton with the sech^2 KdV-like soliton, the expression for u can be represented in a rather simple closed form as follows:

$$\begin{aligned}
&u = \frac{k^2-1}{10((2(k^2-1) + (k^2+2)\cosh\zeta_1)\cosh\zeta_2 - 3k\sinh\zeta_1\sinh\zeta_2)^2}\\
&\qquad \times (k^4 + 11k^2 - 12 + (k^2-4)(\cosh 2\zeta_1 + k^2\cosh 2\zeta_2)\\
&\qquad + 2\cosh\zeta_1(k^4 - 6k^2 + 8 + k^2(2+k^2)\cosh 2\zeta_2) - 6k^3\sinh\zeta_1\sinh 2\zeta_2),\\
&k = 2k_1\sqrt{5}, \qquad \zeta_1 = \frac{k}{\sqrt{5}}\left(x - \left(\frac{k^2}{5} - \frac{k^4}{25}\right)t\right), \qquad \zeta_2 = \frac{1}{\sqrt{5}}\left(x - \frac{4}{25}t\right).
\end{aligned} \tag{31}$$

An example of the two-soliton solution (30) of equation (9) with $\alpha = -1$ corresponding to an overtaking collision of two solitons of the 'exotic' shapes (a) and (c) is shown in figure 2. A head-on collision of two double-hump solitary waves as described by the solution (30) is illustrated by figure 3. An example of the three-soliton solution of equation (9) is presented in figure 4. In the context of the comparison of the GKK solitary wave solutions of equation (9) with solutions of the bidirectional KK equation considered in [13] and [14], it should be noted again that equation (9) with $\alpha = -1$ being a *unidirectional* nonlinear evolution equation nevertheless provides solutions which represent the head-on collisions between solitary waves, as well as overtaking ones.

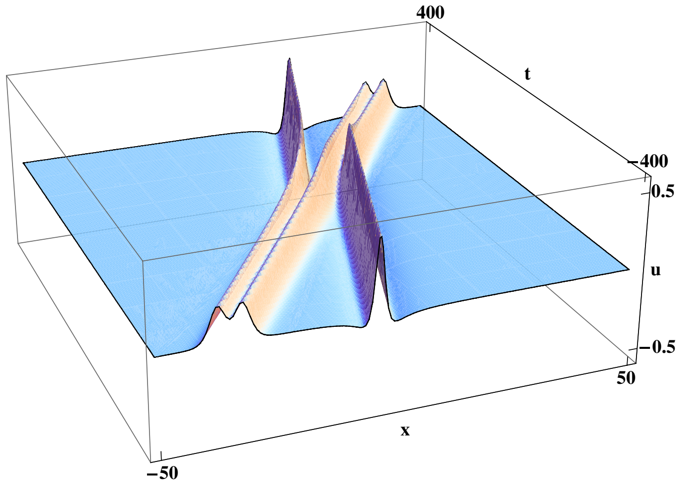

Figure 2. Overtaking collision of the solitary waves of the shapes (*a*) and (*c*) described by the two-soliton solution (30) of equation (9) with $\alpha = -1$. Parameters of the solitons are $Q_1 = -1,\ k_1 = 0.1,\ \phi_1 = 0$ for the first soliton (right, in the front of the picture) and $Q_2 = 1,\ k_2 = 0.475,\ \phi_2 = 0$ for the second soliton (left, in the front of the picture).

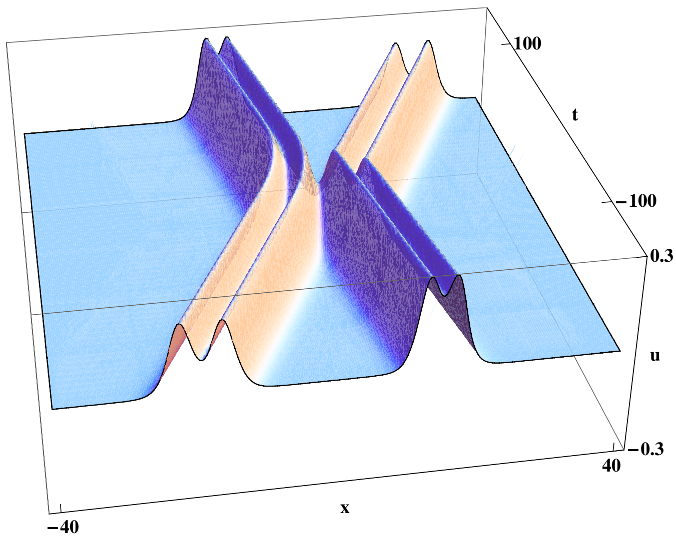

Figure 3. Head-on collision of two double-hump solitons described by the two-soliton solution (30) of equation (9) with $\alpha = -1$. Parameters of the solitons are as follows: $Q_1 = 1,\ k_1 = 0.51,\ \phi_1 = 0$ for the left-running soliton (right, in the front of the picture) and $Q_2 = 1,\ k_2 = 0.46,\ \phi_2 = 0$ for the right-running soliton (left, in the front of the picture).

3.3. Steady-state solutions

The second type of solutions identified by applying the direct method with the Ansatz (22) is the steady-state solutions of the form

$$u = U\frac{1 - kx^2}{(1 + kx^2)^2}, \tag{32}$$

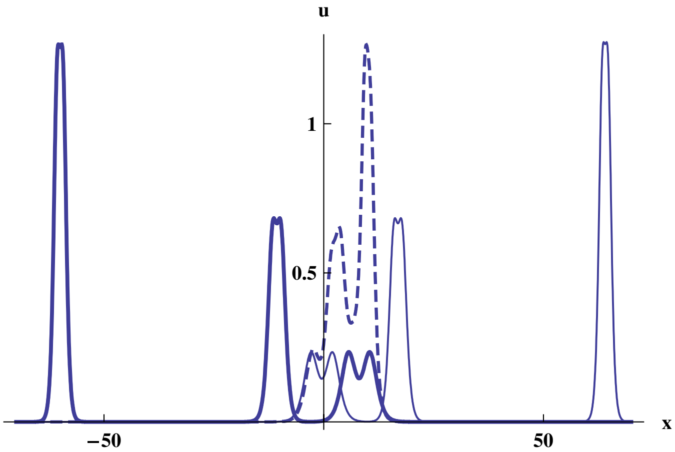

Figure 4. Three-soliton solution of equation (9) with $\alpha = -1$ for the moments $t = -5$ (thin solid), $t = -0.5$ (dashed) and $t = 5$ (thick solid). Parameters of the solitons (according to (10)) are $Q = 1,\ k = 0.475$; $Q = 1,\ k = 0.75$ and $Q = 1,\ k = 1$ for the left, middle and right ones respectively at $t = -5$.

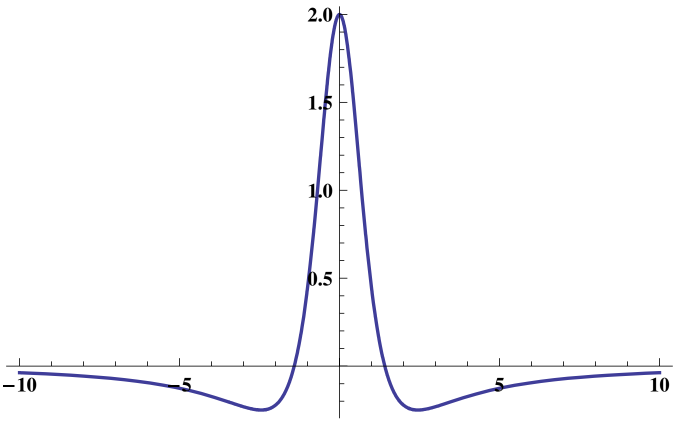

Figure 5. Steady-state solution (32) of the Sawada–Kotera equation (5): $U = 60k/\beta$ for $\beta = 15$ and $k = 0.5$.

where U and k are constants. These solutions describe steady-state localized patterns. Among equations of form (16) only one equation, namely the Sawada–Kotera equation (5), provides such solutions with $U = 60k/\beta$. Note that the shape of the localized pattern described by this solution (figure 5) is qualitatively similar to that of the GKK soliton (25) of type (a) (figure 1) although the former is expressed by an algebraic function (32) while the latter is expressed in terms of hyperbolic functions. Thus, the SK equation possesses two types of solutions describing localized patterns: the sech^2 solitons and steady-state solutions of the form (32). Recall that equation (26), which is one of the two counterparts of (5) among the KdV–SK equations specified to the same value of $\beta = 15$, also possesses solutions of two types: the sech^2 solitary waves and the GKK solitary waves (27) with a velocity not dependent on the shape parameter which, in a sense, is equivalent to the property of being steady state.

Among the equations of type (15) there is a two-parameter family of equations providing solutions of the form (32) for unique k and specific relations between the equation parameters as follows

$$\delta = 45\alpha - \frac{\beta}{2}, \qquad r = 3\beta - 30\alpha, \qquad k = \frac{4}{\beta - 30\alpha}, \qquad U = 2k. \tag{33}$$

The case of $\beta = 30\alpha$ must be excluded and thus the region (33) in the parameter space does not overlap with the region (23) of existence of the KdV-like solitary waves.

It is worth noting here that solutions (32) behave like 'static' solitons when they interact with regular solitons (see remarks in section 4). This is the reason why they are considered in the paper on an equal footing with the common solitary wave solutions.

4. Concluding remarks

In the present paper, new types of the solitary wave solutions of the fifth-order KdV-like equations have been identified. The generalized Kaup–Kupershmidt solitary waves (25) are of most interest. They unify the KdV and KK solitons which become particular cases (or limiting cases as for the solution (10)) of a more general solution. An important finding in this connection is that equation (9) possesses the GKK multi-soliton solutions which makes it a good candidate for being integrable in terms of the GKK solitons. It could be of considerable importance both for asymptotic analysis of the higher order KdV equations and for soliton theory, in general. Another interesting point is that the KdV–SK equation (26), in addition to the sech^2 solitons with a continuous spectrum of the wave velocity, possesses the GKK solitary wave solution with a continuous spectrum of the parameter M defining the soliton shape. Therefore, the solution allows different shapes of solitary waves; in particular, the 'exotic' soliton shapes with two minima and one maximum and two maxima and one minimum are possible. For the GKK solitary wave solutions (10) of equation (9), the shape depends on the parameters α, k and Q. In particular, the 'exotic' shapes arise only in the case of $\alpha = -1$ which corresponds to different signs of the 'physical' parameters $\hat{q}$ and $\hat{\mu}$ in the original equation (11). In this case, solutions may describe both left-running and right-running solitary waves so that the multi-soliton solutions which describe the head-on collisions between solitary waves, as well as overtaking ones, are possible despite the fact that the equation is unidirectional.

Regarding the second solution type which describes the steady-state localized structures, an important point is that they may be considered as (static) solitons. This view is based on the solutions (obtained using a more complicated variant of the method, to be reported separately) describing the interaction of those steady-state localized patterns with regular solitons. It is seen from those solutions that the steady-state structures behave as solitons when they collide with regular solitons—their shape remains unchanged after the collision, only a phase (coordinate) shift is observed.

Some remarks on the method are required here. The solution form using the 'potential' p as an independent variable is a proper tool if the calculations are aimed at defining the solitary wave solutions. An additional advantage of using such variables is that the form of solutions in the original variables is not prescribed from the beginning and, in general, is not directly related to the form of the Ansatz. The above examples show that the calculations starting from the same Ansatz may result in solutions expressed both in terms of hyperbolic functions and in terms of algebraic functions. Such solutions cannot be obtained by applying the popular 'tanh', 'sinh' and so on methods.

Further perspectives are provided first by using other forms of $f(p)$ in the Ansatz (18) and second by enriching the form of the Ansatz, as follows:

$$u(x,t) = f(p(x,t), \xi(x,t)); \qquad p(x,t) = \int u(x,t)\,\mathrm{d}x, \qquad \xi = kx - \omega t - \phi, \tag{34}$$

where the function $f(p, \xi)$ should be such that u vanished for (at least) two values of the 'potential' p in order to get the solitary wave solutions. The above-mentioned solutions describing the interaction of static 'algebraic' solitons with regular 'hyperbolic' solitons have been obtained using the Ansatz (34). Also a version of the method in which p is a dependent variable and u is an independent variable could provide additional possibilities. In such a formulation with $p = F(u)$, one has to solve the equation $u_x = u/F'(u)$ instead of (19).

Just one more remark on the method: it may seem that application of the method is limited to the evolution equations, which can be represented in a conserved form, since an expression for p_t in terms of u and its x-derivatives is used to get a polynomial in p after the substitution of the solution form (18) (or (34)) into the equation. However, it does not limit the application of the method—even if p_t is expressed as an integral (with respect to x) of a combination of u and its x-derivatives; this integral may be transformed into the integral with respect to p and the latter one is easily calculated for specific forms of the Ansatz.

Acknowledgments

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